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Similarity Solutions of the Euler Equation and the Navier–Stokes Equation in Two Space Dimensions

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With the help of the continuous symmetries of the Euler equations and the Navier-Stokes equations, respectively, we derive similarity solutions of these equations for two space dimensions. We show that all group theoretical reductions lead to linear nonautonomous or linear autonomous ordinary differential equations for incompressible fluids.

1. INTRODUCTION

Both the Euler equations and the Navier-Stokes equations are nonlinear systems of partial differential equations and therefore explicit solutions cannot be given in general. However, with the help of the symmetry generators of these equations we can construct similarity Ansätze. Using these similarity Ansätze we can reduce the Euler equations or Navier-Stokes equations to ordinary differential equations which are in general nonlinear. In some cases it is possible to solve these ordinary differential equation. However, in most cases the differential equations must be solved numerically.

In the present paper we give the symmetry generators for the Euler equations and Navier-Stokes equations for two and three space dimensions. Then we use these symmetry generators for constructing similarity Ansätze for the Euler equations and Navier-Stokes equations in two space dimensions, where we assume that the fluid is incompressible. With the similarity Ansätze we derive the ordinary differential equations. We find for incompressible flow that all reductions of the two-dimensional Euler equations leads to linear differential equations.

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In the following we consider the Euler equations and Navier–Stokes equations with dimensionless variables, i.e.,

$$\partial \mathbf{u}/\partial \mathbf{t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \tag{1}$$

and

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + (1/Re)\nabla^2 \mathbf{u} \tag{2}$$

For incompressible flows the dimensionless continuity equation takes the form $\nabla \cdot \mathbf{u} = 0$.

2. INVARIANCE GROUPS

In order to find similarity solutions of an evolution equation we are forced to derive its symmetry group. In the present case Buchnev (1971), Strampp (1980), and Olver (1982) have classified the one-parameter Lie symmetry groups of the Euler equations. The authors give the infinitesimal generators. The technique to find them has been described for example by Bluman and Cole (1974). A different approach has been described by Steeb and Strampp (1982) where also Lie–Bäcklund vector fields (if any exist) can be included. The infinitesimal generators of the Navier–Stokes equations have been given by Lloyd (1981). We mention that if two infinitesimal generators admit a given evolution equation, then the infinitesimal generator which is given by the commutator of these two infinitesimal generators also admit the evolution equation. Consequently, the infinitesimal generators which admit a given evolution equation from a Lie algebra.

The Lie symmetry group of the Euler equations in three space dimensions is generated by the infinitesimal generators ($\mathbf{u} = (u, v, w)$)

$$\begin{aligned}
T &= \partial/\partial t \\
X &= f_1(t)\, \partial/\partial x + f_1'(t)\, \partial/\partial u - f_1''(t) x\, \partial/\partial p \\
Y &= f_2(t)\, \partial/\partial y + f_2'(t)\, \partial/\partial v - f_2''(t) y\, \partial/\partial p \\
Z &= f_3(t)\, \partial/\partial z + f_3'(t)\, \partial/\partial w - f_3''(t) z\, \partial/\partial p \\
S_1 &= x\, \partial/\partial x + y\, \partial/\partial y + z\, \partial/\partial z + t\, \partial/\partial t \\
S_2 &= t\, \partial/\partial t - u\, \partial/\partial u - v\, \partial/\partial v - w\, \partial/\partial w - 2p\, \partial/\partial p
\end{aligned} \tag{3}$$

$$\begin{aligned}
R_{12} &= x\, \partial/\partial y - y\, \partial/\partial x + u\, \partial/\partial v - v\, \partial/\partial u \\
R_{23} &= y\, \partial/\partial z - z\, \partial/\partial y + v\, \partial/\partial w - w\, \partial/\partial v \\
R_{31} &= z\, \partial/\partial x - x\, \partial/\partial z + w\, \partial/\partial u - u\, \partial/\partial w \\
G &= f_4(t)\, \partial/\partial p
\end{aligned} \tag{4}$$

where $f_1, \ldots, f_4$ are arbitrary smooth functions of the time t and $f_i'(t) \equiv df_i(t)/dt$. The physical meaning of these infinitesimal generators is quite obvious. In two space dimension (x, y) the symmetry generators reduce to T, X, Y, R_{12}, G, and

$$\bar{S}_1 = x\,\partial/\partial x + y\,\partial/\partial y + t\,\partial/\partial t$$

$$\bar{S}_2 = t\,\partial/\partial t - u\,\partial/\partial u - v\,\partial/\partial v - 2p\,\partial/\partial p$$

Thus we set $\bar{S} = \bar{S}_1 + \bar{S}_2$. The associated transformation groups which we need for our similarity Ansätze are given by

$$(x, y, z, t, u, v, w, p) \to \exp(\varepsilon K)(x, y, z, t, u, v, w, p) \tag{5}$$

where K stands for one of the infinitesimal generators. For example, consider the infinitesimal generator S_2. Then the associated transformation group is given by

$$(x, y, z, t, u, v, w, p) \to (x, y, z, e^{\varepsilon}t, e^{-\varepsilon}u, e^{-\varepsilon}v, e^{-\varepsilon}w, e^{-2\varepsilon}p) \tag{6}$$

where ε is the group parameter. Frequently, this is written as

$$x = x_0, \qquad y = y_0, \qquad z = z_0, \qquad t = t_0 e^{\varepsilon}$$

$$u = u_0 e^{-\varepsilon}, \qquad v = v_0 e^{-\varepsilon}, \qquad w = w_0 e^{-\varepsilon}, \qquad p = p_0 e^{-2\varepsilon}$$

The Lie symmetry group of the Navier-Stokes equation is generated by the infinitesimal generators T, X, Y, Z, R_{12}, R_{23}, R_{31}, G and $S = S_1 + S_2$ where S generates the one parameter group of scale change. The restriction to two space dimensions is straightforward.

The infinitesimal generator T is associated with the time translation. The infinitesimal generators S, S_1, and S_2 are associated with scale change. Whereas the Euler equations admit two scale transformations, namely, S_1 and S_2, the Navier-Stokes equations admit only one scale transformation, namely, $S = S_1 + S_2$. The infinitesimal generators R_{12}, R_{23}, R_{31} generate a three-parameter rotation group. We mention that the velocities u, v, w rotate with the coordinate. The infinitesimal generators X, Y, and Z are associated with the space translation when we put $f_1(t) = f_2(t) = f_3(t) = 1$. If we take into account an arbitrary f_i, then the moving axes remain parallel to the fixed axes but the origin traces an arbitrary smooth path. The inertial reaction produced by the acceleration of the frame is balanced at each instant by a spatially constant pressure gradient. The transformation group which is associated with the infinitesimal generator G means that the pressure change at each instant is uniform over the fluid and does not affect its motion.

3. SIMILARITY ANSÄTZE FOR FINDING ORDINARY DIFFERENTIAL EQUATIONS

In this section we derive similarity Ansätze for the Euler equations and the Navier–Stokes equations. With these Ansätze we derive a system of ordinary equations.

In our first example we consider the two-dimensional Euler equations and calculate a similarity Ansatz where we take into account the symmetry generators T, $\bar{S}_1$, $\bar{S}_2$, and R_{12}. In this example we describe the technique in detail, whereas in the following examples we give the results only. Since in the present case there are three independent variables we need the symmetry generators which forms the base of a Lie algebra. We choose

$$V_1 = aT + b(\bar{S}_1 - \bar{S}_2) \tag{7a}$$

and

$$V_2 = cR_{12} \tag{7b}$$

where $a, b, c \in \mathbb{R}$. Thus we find

$$V_1 = a\,\partial/\partial t + b(x\,\partial/\partial x + y\,\partial/\partial y + u\,\partial/\partial u + v\,\partial/\partial v + 2p\,\partial/\partial p) \tag{8}$$

We notice that $[V_1, V_2] = 0$.

In order to find the similarity Ansatz we must calculate the transformation groups. Instead of solving equation (5) we can solve the autonomous system of differential equations

$$\begin{aligned} &dt/d\varepsilon_1 = a, \qquad dx/d\varepsilon_1 = bx, \qquad dy/d\varepsilon_1 = by \\ &du/d\varepsilon_1 = bu, \qquad dv/d\varepsilon_1 = bv, \qquad dp/d\varepsilon_1 = 2bp \end{aligned} \tag{9}$$

leads to the transformation group

$$\begin{aligned} &t = a\varepsilon_1 + t_0, \qquad x = x_0\,e^{b\varepsilon_1}, \qquad y = y_0\,e^{b\varepsilon_1} \\ &u = u_0\,e^{b\varepsilon_1}, \qquad v = v_0\,e^{b\varepsilon_1}, \qquad p = p_0\,e^{2b\varepsilon_1} \end{aligned} \tag{10}$$

where ε_1 is the group parameter. With the infinitesimal generator V_2 the autonomous system

$$\begin{aligned} &dx/d\varepsilon_2 = -cy, \qquad dy/d\varepsilon_2 = cx \\ &du/d\varepsilon_2 = -cv, \qquad dv/d\varepsilon_2 = cu \end{aligned} \tag{11}$$

is associated. We obtain the transformation group

$$\begin{aligned} x &= x_0 \cos c\varepsilon_2 - y_0 \sin c\varepsilon_2 \\ y &= y_0 \cos c\varepsilon_2 + x_0 \sin c\varepsilon_2 \\ u &= u_0 \cos c\varepsilon_2 - v_0 \sin c\varepsilon_2 \\ v &= v_0 \cos c\varepsilon_2 + u_0 \sin c\varepsilon_2 \end{aligned} \tag{12}$$

The composition of both groups leads to the two-parameter group

$$t = a\varepsilon_1 + t_0 \tag{13a}$$

$$x = [x_0 \cos(c\varepsilon_2) - y_0 \sin(c\varepsilon_2)]\, e^{b\varepsilon_1} \tag{13b}$$

$$y = [y_0 \cos(c\varepsilon_2) + x_0 \sin(c\varepsilon_2)]\, e^{b\varepsilon_1} \tag{13c}$$

$$u = [u_0 \cos(c\varepsilon_2) - v_0 \sin(c\varepsilon_2)]\, e^{b\varepsilon_1} \tag{14a}$$

$$v = [v_0 \cos(c\varepsilon_2) + u_0 \sin(c\varepsilon_2)]\, e^{b\varepsilon_1} \tag{14b}$$

$$p = p_0\, e^{2b\varepsilon_1} \tag{14c}$$

For finding a similarity variable η we have to specify the variables x_0, y_0, and t_0. We choose $x_0 = 1$, $y_0 = 0$, and $t_0 = \eta$. From equation (13) it follows that

$$\begin{aligned} t &= a\varepsilon_1 + \eta \\ x &= e^{b\varepsilon_1} \cos(c\varepsilon_2) \\ y &= e^{b\varepsilon_1} \sin(c\varepsilon_2) \end{aligned} \tag{15}$$

The solution of these equations is given by

$$\begin{aligned} &\varepsilon_1 = (1/b) \ln r, \qquad \varepsilon_2 = (1/c) \arctan(y/x) \\ &\eta = t - (a/b) \ln r \end{aligned} \tag{16}$$

where $r^2 = x^2 + y^2$. Inserting ε_1, ε_2, and η into the equation (14) we obtain the similarity Ansatz

$$\begin{aligned} u(x, y, t) &= x u_0(\eta) - y v_0(\eta) \\ v(x, y, t) &= x v_0(\eta) + y u_0(\eta) \\ p(x, y, t) &= r^2 p_0(\eta) \end{aligned} \tag{17}$$

Inserting this similarity Ansatz into the Euler equations (1) we find after some algebraic manipulation the autonomous system of differential equations

$$u_0' = (2b/a) u_0 \tag{18a}$$

$$v_0'[1 - (a/b) u_0] = -2 u_0 v_0 \tag{18b}$$

$$p_0' = -(\rho b/a)(u_0^2 + v_0^2) + 2\rho (b/a)^2 u_0 + (2b/a) p_0 \tag{18c}$$

where $' \equiv d/d\eta$. Equation (18a) follows at once from the continuity equation.

In our next example we consider the Navier-Stokes equations in two space dimensions and the symmetry generators T, X, and Y where $f_1 = \text{const}$

and $f_2 = \text{const}$. By inspection this leads to the similarity Ansatz

$$\begin{aligned} u(x, y, t) &= u_0(\eta) \\ v(x, y, t) &= v_0(\eta) \\ p(x, y, t) &= p_0(\eta) \end{aligned} \tag{19}$$

where the similarity variable η is given by

$$\eta = ax + by + ct \tag{20}$$

Inserting this Ansatz into the Navier-Stokes equations we find after some algebraic manipulation the system of ordinary equations

$$u_0'' = \frac{Re}{a^2+b^2}(c + au_0' + bv_0')u_0' \tag{21a}$$

$$v_0'' = \frac{Re}{a^2+b^2}(c + au_0' + bv_0')v_0' \tag{21b}$$

$$p_0' = 0 \tag{21c}$$

From the continuity equation we obtain $au_0' + bv_0' = 0$. It follows that $au_0 + bv_0 = K$, where K is a constant. Therefore equations (21a) and (21b) reduce to a linear system of differential equations

$$\begin{aligned} u_0'' &= \frac{Re}{a^2+b^2}(c+K)u_0' \\ v_0'' &= \frac{Re}{a^2+b^2}(c+K)v_0' \end{aligned} \tag{22}$$

In our third example we consider the infinitesimal generators aR_{12} and bS. We mention that $[R_{12}, \bar{S}] = 0$ and therefore these infinitesimal generators can be used for deriving a similarity Ansatz. We find the two-parameter transformation group is given by

$$\begin{aligned} x &= x_0\, e^{b\varepsilon_1} \cos(a\varepsilon_2) - y_0\, e^{b\varepsilon_1} \sin(a\varepsilon_2) \\ y &= y_0\, e^{b\varepsilon_1} \cos(a\varepsilon_2) + x_0\, e^{b\varepsilon_1} \sin(a\varepsilon_2) \\ t &= t_0\, e^{2b\varepsilon_1} \end{aligned} \tag{23}$$

$$\begin{aligned} u &= u_0\, e^{-b\varepsilon_1} \cos(a\varepsilon_2) - v_0\, e^{-b\varepsilon_1} \sin(a\varepsilon_2) \\ v &= v_0\, e^{-b\varepsilon_1} \cos(a\varepsilon_2) + u_0\, e^{-b\varepsilon_1} \sin(a\varepsilon_2) \\ p &= p_0\, e^{-2b\varepsilon_1} \end{aligned} \tag{24}$$

If we choose $t_0 = \eta$, $x_0 = 1$, and y_0, then we find that $\varepsilon_1 = (1/2b)\ln r^2$, $\varepsilon_2 = (1/a)\arctan(y/x)$ and $\eta = t/r^2$, where $r^2 = x^2 + y^2$. The similarity

Ansatz is given by

$$
\begin{aligned}
u(x, y, t) &= xu_0(\eta)/r^2 - yv_0(\eta)/r^2 \\
v(x, y, t) &= xv_0(\eta)/r^2 + yu_0(\eta)/r^2 \\
p(x, y, t) &= p_0(\eta)/r^2
\end{aligned} \tag{25}
$$

Substituting this Ansatz into equation (2) we find the following system of ordinary differential equations:

$$u_0' = 0 \tag{26a}$$

$$v_0'' + v_0'\left(\frac{2}{\eta} - \frac{Re}{4\eta^2} + \frac{Re}{2\eta}u_0\right) = 0 \tag{26b}$$

$$p_0' + \frac{1}{\eta}p_0 + \frac{1}{2\eta}(u_0^2 + v_0^2) = 0 \tag{26c}$$

Equation (26a) follows at once from the continuity equation.

In our last example we study two-dimensional Navier–Stokes equation and the symmetry generators $a(X+Y)$ and cS, where $a, c \in \mathbb{R}$. We note that $[aX + aY, cS] = ac(X+Y)$ and therefore a similarity Ansatz can be found with the help of these generators.

If we insert $t_0 = \eta$, $x_0 = 1$, and $y_0 = 0$, in the two-parameter transformation group, then we find $\varepsilon_1 = y(ax - ay)$, $\varepsilon_2 = (1/c)\ln(x-y)$

$$\eta = \frac{t}{(x-y)^2} \tag{27}$$

The similarity Ansatz is given by

$$
\begin{aligned}
u(x, y, t) &= \frac{u_0(\eta)}{x-y} \\
v(x, y, t) &= \frac{v_0(\eta)}{x-y} \\
p(x, y, t) &= \frac{p_0(\eta)}{(x-y)^2}
\end{aligned} \tag{28}
$$

Inserting this Ansatz into the continuity equation we find that

$$2\eta(u_0 - v_0)' + (u_0 - v_0) = 0 \tag{29}$$

This leads to

$$u_0 - v_0 = C\eta^{-1/2} \tag{30}$$

Inserting equation (28) into equation (2) and taking into account equation (30) we obtain

$$\eta^3 u_0'' - \left(\eta\delta - 2\delta\eta^{3/2}C - \frac{5}{2}\eta^2\right)u_0' + \left(\frac{\eta}{2} + \delta\eta^{1/2}C\right)u_0 - \frac{\delta}{4}C\eta^{-1/2} = 0$$

$$\eta^3 v_0'' - \left(\eta\delta - 2\delta\eta^{3/2}C - \frac{5}{2}\eta^2\right)v_0' + \left(\frac{\eta}{2} + \delta\eta^{1/2}C\right)v_0 + \frac{\delta}{4}C\eta^{-1/2} = 0 \qquad (31)$$

$$8\eta^2 p_0' + 8\eta p_0 + C\eta^{-1/2} = 0$$

where $\delta = Re/8$. For the particular case where $C = 0$ we find

$$\eta^3 u_0'' - \left(\eta\delta - \frac{5}{2}\eta^2\right)u_0' + \frac{\eta}{2}u_0 = 0 \qquad (32a)$$

$$\eta^3 v_0'' - \left(\eta\delta - \frac{5}{2}\eta^2\right)v_0' + \frac{\eta}{2}v_0 = 0 \qquad (32b)$$

$$\eta p_0' + p_0 = 0 \qquad (32c)$$

If $\delta = 0$ (i.e., $Re = 0$), then the solutions to equations (32a) and (32b) are given by

$$u_0(\eta) = C_1\eta^{-1/2} + C_2\eta^{-1}$$
$$v_0(\eta) = C_3\eta^{-1/2} + C_4\eta^{-1} \qquad (33)$$

Also the asymptotic behavior of equations (32a) and (32b) (i.e., $\eta \to \infty$) is given by these expressions.

The Euler equations and the Navier-Stokes equations given by equations (1) and (2) together with equation (3) can be studied for two space dimensions from a different point of view introducing the so-called stream function ψ. The stream function ψ is given by

$$u = \frac{\partial\psi}{\partial x}, \qquad v = -\frac{\partial\psi}{\partial y} \qquad (34)$$

Eliminating the pressure p in equation (2) by differentiation and taking into account equation (3) we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - \frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial x^2\,\partial y}\right) + \frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\,\partial y^2}\right)$$
$$-\frac{1}{Re}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + 2\frac{\partial^4\psi}{\partial x^2\,\partial y^2}\right) = 0 \qquad (35)$$

If we put $Re \to \infty$, then we obtain the equation which is associated with the Euler equation (1), namely,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y^3}+\frac{\partial^3\psi}{\partial x^2\,\partial y}\right)+\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x\,\partial y^2}\right)=0 \tag{36}$$

The Lie symmetry groups of the equations (34) and (35) have been given by Cantwell (1978). The infinitesimal generators take the form

$$\begin{aligned}
T &= \partial/\partial t\\
X &= f_1(t)\,\partial/\partial t+f_1'(t)y\,\partial/\partial\psi\\
Y &= f_2(t)\,\partial/\partial y-f_2'(t)x\,\partial/\partial\psi\\
G &= f_3(t)\,\partial/\partial\psi\\
S &= x\,\partial/\partial x+y\,\partial/\partial y+kt\,\partial/\partial t+(2-k)\psi\,\partial/\partial\psi\\
R_{12} &= x\,\partial/\partial y-y\,\partial/\partial x\\
R &= ty\,\partial/\partial x-tx\,\partial/\partial x+(1/2)(x^2+y^2)\,\partial/\partial\psi
\end{aligned} \tag{37}$$

where k is arbitrary for equation (36) and $k=2$ for equation (35).

Let us now consider group theoretical reductions of equation (36). Taking into account space and time translation X, Y, and T with $f_1(t)=1$ and $f_2(t)=1$ we obtain

$$\psi(x, y, t)=f(k_1x+k_2y-\omega t) \tag{38}$$

Inserting equation (38) into equation (36) yields

$$\omega(k_1^2+k_2^2)f'''=0 \tag{39}$$

The group theoretical reduction with the help of S (with $k=2$) and $X+Y$ [with $f_1(t)=f_2(t)=1$] yields

$$\psi(x, y, t)=f(\eta) \tag{40}$$

where

$$\eta=t/(x-y)^2 \tag{41}$$

We obtain

$$2\eta^2f'''+7\eta f''+3f'=0 \tag{42}$$

The group theoretical reduction with the help of K and R_{12} yields

$$\psi(x, y, t)=f(\eta) \tag{43}$$

where

$$\eta=(x^2+y^2)^{1/2}/t^{1/2} \tag{44}$$

We obtain

$$\eta^2 f''' + 3\eta f'' + f' = 0 \tag{45}$$

Also all other reductions give linear differential equations. For the equation (35) the reduction also yields linear differential equations. Notice that also equation (18) is linear, because equation (18a) is linear and the solution can be inserted into equation (18b) which becomes then also linear but nonautonomous. Then inserting the solutions to equations (18a) and (18b) we find that the equations (18c) also become linear.

If we take into account only one symmetry generator, then we can reduce the partial differential equation (36) with three independent variables to a partial differential equation with two independent variables. Let us study three cases. If we consider the symmetry generator given by equation (38), then we find the Ansatz

$$\psi(x, y, t) = f(r, t) \tag{46}$$

where $r^2 = x^2 + y^2$. Inserting equation (46) into equation (36) we find the linear partial differential equation

$$\partial^3\psi/\partial t\, \partial r^2 + (1/r)\, \partial^2\psi/\partial t\, \partial r = 0 \tag{47}$$

If we consider the symmetry generator $X + Y$, then we have

$$\psi(x, y, t) = f(k_1 x + k_2 y, t) \tag{48}$$

We obtain again a linear partial differential equation

$$\partial^3 f/\partial t\, \partial\eta^2 = 0$$

where $\eta = k_1 x + k_2 y$. The following case has been studied by Cantwell (1978). He considered the invariance under stretching. The infinitesimal generator is given by

$$S = x\, \partial/\partial x + y\, \partial/\partial y + kt\, \partial/\partial t + (2-k)\psi\, \partial/\partial\psi \tag{49}$$

where k is an arbitrary constant. The similarity variables are given by

$$\Theta = xt^{-1/k}, \qquad \phi = yt^{-1/k} \tag{50}$$

and the similarity Ansatz takes the form

$$\psi(x, y, t) = t^{(2-k)/k} F(\Theta, \phi) \tag{51}$$

The function F satisfies a nonlinear partial differential equation, namely,

$$-\Delta F + (F_\phi - (\Theta/2))\, \Delta F_\Theta - (F_\Theta + (\phi/2))\, \Delta F_\phi = (1/Re)\, \Delta^2 F \tag{52}$$

where $\Delta = \partial^2/\partial\Theta^2 + \partial^2/\partial\phi^2$ and $k = 2$.

4. CONCLUSION

For the Navier–Stokes equations and the Euler equations we have given the infinitesimal generators for two and three space dimensions, where we have assumed that the fluid is incompressible. The utility of the infinitesimal generators is twofold. First we can find conservation laws. For the Euler equations this program has been performed in detail by Olver (1982). Second as described above the infinitesimal generators can be used for finding similarity Ansätze. With the help of these similarity Ansätze we can reduce the partial differential equations to ordinary differential equations or to partial differential equations where the number of the independent variables is lower. For the present case we always find that the ordinary differential equations are linear, but in most cases non-autonomous. Although these equations are linear, we are not able in general to solve them explicitly. For example, equations (22) can be solved explicitly; however, equations (31) cannot be solved explicitly. We are forced to solve this equation numerically.

It is obvious that the similarity solutions are particular solutions. This means, that if we impose boundary conditions, then it may happen that these conditions are not compatible with the similarity Ansatz.

Since all reductions lead to linear equations the question arises whether the equation (37) is integrable (i.e., a soliton equation) or not. Particular Ansätze for Lie–Bäcklund vector fields for the equation (37) are not successful. Thus we conjugate that the equation (37) is not integrable.

REFERENCES

Bluman, G. W., and Cole, J. D. (1974). *Similarity Methods for Differential Equations, Appl. Mathe. Sci., Vol. 13* (Springer, New York).

Buchnev, A. A. (1971). *Dinamika Splosn. Sredy,* **7**, 212.

Cantwell, B. J. (1978). *J. Fluid Mech.*, **85**, 257.

Lloyd, S. P. (1981). *Acta Mechanica,* **38**, 85.

Olver, P. J. (1982). *J. Math. Anal. Appl.*, **89**, 233.

Steeb, W.-H., and Strampp, W. (1982). *Physica,* **114A**, 95.

Strampp, W. (1980). *J. Math. Anal. Appl.*, **78**, 618.